

INTEGRAL FINITE SURGERIES ON KNOTS IN S^3

LILING GU

ABSTRACT. Using the correction terms in Heegaard Floer homology, we prove that if a knot in S^3 admits a positive integral **T**-, **O**- or **I**-type surgery, it must have the same knot Floer homology as one of the knots given in our complete list, and the resulting manifold is orientation-preservingly homeomorphic to the p -surgery on the corresponding knot.

1. INTRODUCTION

In the early 1960s, Wallace [27] and Lickorish [11] proved independently that any closed, orientable, connected 3-manifold can be obtained by performing Dehn surgery on a framed link in the 3-sphere. One natural question is which manifolds can be obtained by some surgery on a knot. In this paper we consider the manifolds with finite noncyclic fundamental groups. By Perelman's resolution of the Geometrization Conjecture [21, 22, 23], the manifolds with finite fundamental group are spherical space forms. They fall into five classes, those with cyclic π_1 and those with finite π_1 based on the four isometries of a sphere:

Theorem 1.1. (*Seifert [25]*). *If Y^3 is closed, oriented and Seifert-fibered with finite but noncyclic fundamental group, then it has base orbifold S^2 and is one of:*

- (1) *Type **D**, dihedral:* $(b; \frac{1}{2}, \frac{1}{2}, \frac{a_3}{b_3})$
- (2) *Type **T**, tetrahedral:* $(b; \frac{1}{2}, \frac{a_2}{3}, \frac{a_3}{3})$
- (3) *Type **O**, octahedral:* $(b; \frac{1}{2}, \frac{a_2}{3}, \frac{a_3}{4})$
- (4) *Type **I**, icosahedral:* $(b; \frac{1}{2}, \frac{a_2}{3}, \frac{a_3}{5})$

with coefficients chosen so that $(a_i, b_i) = 1$.

Berge [1] constructed a list of knots which yield lens space surgeries, and he conjectured that it is complete. Greene [9] proved that if a p -surgery along a knot produces a lens space $L(p, q)$, there exists a p -surgery along a Berge knot with the same knot Floer homology groups as $L(p, q)$. In this paper we focus on knots with finite noncyclic surgeries. According to Thurston [26], a knot is either a torus knot, a hyperbolic knot, or a satellite knot. Moser [13] classified all finite surgeries on torus knots, and Bleiler and Hodgson [3] classified all finite surgeries on cables of torus knots. Boyer and Zhang [4] showed that a satellite knot with a finite noncyclic surgery must be a cable of some torus knot. There is also some progress about hyperbolic knots. Doig [7] proved that there are only finitely many spherical space forms which come from a p/q -surgery on S^3 for a fixed integer p . From now on we will only consider Dehn surgeries on hyperbolic knots.

Suppose M is a 3-manifold with torus boundary, α is a slope on ∂M . Let $M(\alpha)$ be the Dehn filling along α . If M is hyperbolic, Thurston's Hyperbolic Dehn Surgery Theorem says that at most finitely many of the fillings are nonhyperbolic. These surgeries are called *exceptional surgeries*. In [4], Boyer and Zhang showed

that if M is hyperbolic, $M(\alpha)$ has a finite fundamental group and $M(\beta)$ has a cyclic fundamental group, then $|\Delta(\alpha, \beta)| \leq 2$. In particular, if the p/q -surgery on a hyperbolic knot $K \subset S^3$, denoted $S_K^3(p/q)$, has a finite fundamental group, then $|q| \leq 2$. For $|q| = 2$, Li and Ni [10] prove that K has the same knot Floer homology as either $T(5, 2)$ or a cable of a torus knot (which must be $T(3, 2)$ or $T(5, 2)$). From now on we will only consider integral surgeries on hyperbolic knots.

Taking the mirror image of a knot K if necessary, we may assume $p > 0$. We consider here all **T**-, **O**- and **I**-type spherical space forms. In this paper, all manifolds are oriented. If Y is an oriented manifold, then $-Y$ denotes the same manifold with the opposite orientation. Let \mathbb{T} be the exterior of the right-hand trefoil, then $\mathbb{T}(p/q)$ is the manifold obtained by p/q -surgery on the right hand trefoil. It is well-known that any **T**-, **O**-, or **I**-type manifold is homeomorphic to a $\pm\mathbb{T}(p/q)$ (see Lemma 3.2).

Our main result is the following theorem.

Theorem 1.2. *Suppose that K is a knot in S^3 , and that the p -surgery on K is a **T**-, **O**- or **I**-type spherical space form for some integer $p > 0$, then K has the same knot Floer homology as one of the knots \tilde{K} in table 3 or table 4, and the resulting manifold is orientation-preservingly homeomorphic to the p -surgery on the corresponding knot \tilde{K} .*

The strategy of our proof is to compute the Heegaard Floer correction terms for the **T**-, **O**- and **I**-type manifolds, and to compare them with the correction terms of the integral surgeries on knots in S^3 . If they match, the knot Floer homology of the knots can be recovered from the correction terms. Knot Floer homology contains a lot of information about knots. For example, it detects the genus [17] and fiberedness [14]. We propose the following conjecture.

Conjecture 1.3. *Suppose that K is a hyperbolic knot in S^3 , and that the p -surgery on K is a **T**-, **O**- or **I**-type spherical space form for some integer $p > 0$, then K is one of the knots in table 4.*

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2. PRELIMINARIES ON HEEGAARD FLOER HOMOLOGY AND CORRECTION TERMS

Heegaard Floer homology was introduced by Ozsváth and Szabó [16]. Given a closed oriented 3-manifold Y and a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(Y)$, one can define the Heegaard Floer homology $\widehat{HF}(Y, \mathfrak{s}), HF^+(Y, \mathfrak{s}), \dots$, which are invariants of (Y, \mathfrak{s}) . When \mathfrak{s} is torsion, there is an absolute \mathbb{Q} -grading on $HF^+(Y, \mathfrak{s})$. When Y is a rational homology sphere, Ozsváth and Szabó [15] defined a *correction term* $d(Y, \mathfrak{s}) \in \mathbb{Q}$, which is the shifting of the absolute grading of $HF^+(Y, \mathfrak{s})$ relative to $HF^+(S^3, 0)$.

The correction terms have the following symmetries:

$$(1) \quad d(Y, \mathfrak{s}) = d(Y, J\mathfrak{s}), \quad d(-Y, \mathfrak{s}) = -d(Y, \mathfrak{s}),$$

where $J : \text{Spin}^c(Y) \rightarrow \text{Spin}^c(Y)$ is the conjugation.

Suppose that Y is an integral homology sphere, $K \subset Y$ is a knot. Let $Y_K(p/q)$ be the manifold obtained by p/q -surgery on K . Ozsváth and Szabó defined a natural

identification $\sigma : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Spin}^c(Y_K(p/q))$ [15, 20]. For simplicity, we often use an integer i to denote the Spin^c structure $\sigma([i])$, when $[i] \in \mathbb{Z}/p\mathbb{Z}$ is the congruence class of i modulo p .

A rational homology sphere Y is an L -space if $\text{rank} \widehat{HF}(Y) = |H_1(Y)|$. Examples of L -spaces include spherical space forms. The information about the Heegaard Floer homology of an L -space is completely encoded in its correction terms.

Let $L(p, q)$ be the lens space obtained by p/q -surgery on the unknot. The correction terms for lens spaces can be computed inductively as follows:

$$d(S^3, 0) = 0,$$

$$(2) \quad d(-L(p, q), i) = \frac{1}{4} - \frac{(2i+1-p-q)^2}{4pq} - d(-L(q, r), j)$$

where $0 \leq i < p+q$, r and j are the reductions of p and i modulo q , respectively.

For example, using the recursion formula (2), we get

$$(3) \quad d(L(3, q), i) = \begin{cases} (\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}) & q = 1, i = 0, 1, 2 \\ (\frac{1}{6}, \frac{1}{6}, -\frac{1}{2}) & q = 2, i = 0, 1, 2 \end{cases}$$

$$(4) \quad d(L(4, q), i) = \begin{cases} (\frac{3}{4}, 0, -\frac{1}{4}, 0) & q = 1, i = 0, 1, 2, 3 \\ (0, \frac{1}{4}, 0, -\frac{3}{4}) & q = 3, i = 0, 1, 2, 3 \end{cases}$$

$$(5) \quad d(L(5, q), i) = \begin{cases} (1, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \frac{1}{5}) & q = 1, i = 0, 1, 2, 3, 4 \\ (\frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, 0, -\frac{2}{5}) & q = 2, i = 0, 1, 2, 3, 4 \\ (\frac{3}{5}, 0, \frac{2}{5}, -\frac{2}{5}, -\frac{3}{5}) & q = 3, i = 0, 1, 2, 3, 4 \\ (-\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -1) & q = 4, i = 0, 1, 2, 3, 4 \end{cases}$$

Given a null-homologous knot $K \subset Y$, Ozsváth and Szabó [18] and Rasmussen [24] defined the knot Floer homology. From [20], if we know the knot Floer homology, then we can compute the Heegaard Floer homology of all the surgeries on K . In particular, if the p/q -surgery on $K \subset S^3$ is an L -space surgery, where $p, q > 0$, then the correction terms of $S_K^3(p/q)$ can be computed from the Alexander polynomial of K as follows.

Suppose

$$\Delta_K(T) = a_0 + \sum_{i>0} a_i(T^i + T^{-i})$$

Define a sequence of integers

$$t_i = \sum_{j=1}^{\infty} j a_{i+j}, \quad i \geq 0.$$

then a_i can be recovered from t_i by

$$(6) \quad a_i = t_{i-1} - 2t_i + t_{i+1}, \quad \text{for } i > 0.$$

If K admits an L -space surgery, then one can prove [20, 24]

$$(7) \quad t_s \geq 0, \quad t_s \geq t_{s+1} \geq t_s - 1, \quad t_{g(K)} = 0.$$

Moreover, the following proposition holds.

Proposition 2.1. *Suppose the p/q -surgery on $K \subset S^3$ is an L -space surgery, where $p, q > 0$. Then for any $0 \leq i < p$ we have*

$$d(S_K^3(p/q), i) = d(L(p, q), i) - 2 \max\{t_{\lfloor \frac{i}{q} \rfloor}, t_{\lfloor \frac{p+q-1-i}{q} \rfloor}\}.$$

This formula is contained in Ozsváth and Szabó [20] and Rasmussen [24].

3. THE STRATEGY OF OUR PROOF

Recall that $\phi : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Spin}^c(Y_K(p/q))$ is the natural identification defined by Ozsváth and Szabó. The following two lemmas are from [10].

Lemma 3.1. *Suppose i is an integer satisfying $0 \leq i < p + q$, then $J(\sigma([i]))$ is represented by $p + q - 1 - i$.*

Lemma 3.2. *Any **T**-type manifold is homeomorphic to $\pm \mathbb{T}(\frac{6q \pm 3}{q})$ for some positive integer q with $(q, 3) = 1$. Any **O**-type manifold is homeomorphic to $\pm \mathbb{T}(\frac{6q \pm 4}{q})$ for some positive integer q with $(q, 2) = 1$. Any **I**-type manifold is homeomorphic to $\pm \mathbb{T}(\frac{6q \pm 5}{q})$ for some positive integer q with $(q, 5) = 1$.*

Let $p, q > 0$ be coprime integers. Using Proposition 2.1, we get

$$(8) \quad d(\mathbb{T}(p/q), i) = d(L(p, q), i) - 2\chi_{[0, q)}(i),$$

$$\text{where } \chi_{[0, q)}(i) = \begin{cases} 1 & \text{when } 0 \leq i < q \\ 0 & \text{when } q \leq i < p \end{cases}$$

Suppose $S_K^3(p)$ is a spherical space form, then by Proposition 2.1,

$$\begin{aligned} d(S_K^3(p), i) &= d(L(p, 1), i) - 2\max\{t_i, t_{p-i}\} \\ &= \frac{(2i - p)^2 - p}{4p} - 2t_{\min(i, p-i)} \end{aligned}$$

If $S_K^3(p) \cong \varepsilon \mathbb{T}(p/q)$, $\varepsilon = \pm 1$, then the two sets

$$\{d(S_K^3(p), i) | i \in \mathbb{Z}/p\mathbb{Z}\}, \quad \{\varepsilon d(\mathbb{T}(p/q), i) | i \in \mathbb{Z}/p\mathbb{Z}\}$$

are equal. However, the two parametrizations of Spin^c structures may differ by an affine isomorphism of $\mathbb{Z}/p\mathbb{Z}$. More precisely, there exists an affine isomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$, such that

$$d(S_K^3(p), i) = \varepsilon d(\mathbb{T}(p/q), \phi(i)).$$

For any integers a, b , define $\phi_{a,b} : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ by

$$\phi_{a,b}(i) = ai + b \pmod{p}$$

Lemma 3.3. *There are at most two values for b , $b_j = \frac{jp+q-1}{2}$, $j = 0, 1$.*

Proof. The affine isomorphism ϕ commutes with J , i.e., $\phi J_p = J_{\frac{p}{q}} \phi$. Using lemma 3.1, we get the desired values for b . Note that b_0 or b_1 may be a half-interger, in this case we discard it. \square

Note $\phi_{a,b}(i) = \phi_{p-a,b}(p-i)$, By (1) and Lemma 3.1,

$$d(\mathbb{T}(p/q), \phi_{a,b}(i)) = d(\mathbb{T}(p/q), \phi_{p-a,b}(p-i)) = d(\mathbb{T}(p/q), \phi_{p-a,b}(i)).$$

So we may assume

$$(9) \quad 0 < a < \frac{p}{2}, \quad (p, a) = 1.$$

Then we may assume

$$d(S_K^3(p), i) = \varepsilon d(\mathbb{T}(p/q), \phi_{a,b_j}(i)), \quad \text{for some } a, \text{ any } i \in \mathbb{Z}/p\mathbb{Z}, \text{ and } j = 0 \text{ or } 1.$$

Let

$$(10) \quad \Delta_{a,b_j}^\varepsilon(i) = d(L(p, 1), i) - \varepsilon d(\mathbb{T}(p/q), \phi_{a,b_j}(i))$$

By Proposition 2.1, we should have

$$(11) \quad \Delta_{a,b_j}^\varepsilon(i) = 2t_{\min(i, p-i)}$$

if $S_K^3(p) \cong \varepsilon \mathbb{T}(p/q)$ and ϕ_{a,b_j} identifies their Spin^c structures.

In order to prove Theorem 1.2, we will compute the correction terms of the **T**-, **O**- and **I**-type manifolds using (8). For all a satisfying (9), we compute the sequences $\Delta_{a,b_j}^\varepsilon(i)$. Then we check whether they satisfy (11) for some $\{t_s\}$ as in (7). We will show that (11) cannot be satisfied when p is sufficiently large. For small p , a direct computation yields all the p/q 's. By a standard argument in Heegaard Floer homology [19], we can get the knot Floer homology of the corresponding knots, which should be the knot Floer homology of either a (p, q) -torus knot $((p, q) = (2, 3), (2, 5), (3, 4), (3, 5))$, a cable knot or some hyperbolic knot. We will list torus knots and cables of torus knots separately for completeness, one may also consult Moser [13] and Bleiler and Hodgson [3].

4. THE CASE WHEN p IS LARGE

In this section, we will assume that $S_K^3(p) \cong \varepsilon \mathbb{T}(p/q)$, and

$$p = 6q + \zeta r, r \in \{3, 4, 5\}, \varepsilon, \zeta \in \{-1, 1\}.$$

We will prove that this does not happen when p is sufficiently large.

Proposition 4.1. *If $p > 310r(36r + 1)^2$, then $S_K^3(p) \not\cong \varepsilon \mathbb{T}(p/q)$, where $p = 6q + \zeta r, r \in \{3, 4, 5\}$.*

Let $s \in \{0, 1, \dots, r-1\}$ be the reduction of q modulo r . For any integer n , let $\theta(n) \in \{0, 1\}$ be the reduction of n modulo 2, and let $\bar{\theta}(n) = 1 - \theta(n)$.

Lemma 4.2. *For $0 \leq i < q$,*

$$(12) \quad d(L(q, \frac{1-\zeta}{2}q + \zeta r), i) = \zeta \left(\frac{(2i+1-q-\zeta r)^2}{4qr} - \frac{1}{4} - d(L(r, s), i \mod r) \right).$$

Proof. For $0 \leq i < q$, using (2), we have

$$\begin{aligned} d(L(q, r), i) &= \frac{(2i+1-q-r)^2}{4qr} - \frac{1}{4} - d(r, s, i \mod r) \\ d(L(q, q-r), i) &= \frac{(2i+1-2q+r)^2}{4q(q-r)} - \frac{1}{4} - d(L(q-r, r), i) \\ &= \frac{(2i+1-2q+r)^2}{4q(q-r)} - \frac{(2i+1-q)^2}{4r(q-r)} + d(L(r, s), i \mod r) \\ &= - \left(\frac{(2i+1-q+r)^2}{4qr} - \frac{1}{4} - d(L(r, s), i \mod r) \right) \end{aligned}$$

□

Recall $\phi_{a,b} : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ is defined by

$$\phi_{a,b}(i) = ai + b \pmod{p}.$$

Lemma 4.3. *When $p > 52$, there is at most one value for b in $\{b_0, b_1\}$.*

Proof. For a **T**- or **I**-type p -surgery on a knot, by lemma 3.2, $S_K^3(p) \cong \varepsilon\mathbb{T}(p/q)$, where $p = 6q + \zeta r$, $r = 3$ or 5 . Here p is odd, $\frac{p}{2}$ is a half integer. By lemma 3.3, if q is odd, $b = \frac{q-1}{2}$; if q is even, $b = \frac{p+q-1}{2}$. We may write $b = \frac{\bar{\theta}(q)p+q-1}{2}$.

For an **O**-type p -surgery on a knot, by lemma 3.2, $S_K^3(p) \cong \varepsilon\mathbb{T}(p/q)$, where $p = 6q + \zeta r$, $r = 4$. Note here p is even, $(p, q) = (p, a) = 1$, q, a are odd, so $q = 4l + s$, where $s = 1, 3$. By lemma 3.3, $b_j = \frac{jp+q-1}{2}$, $j = 0, 1$, and both of them are integers. Denote $\phi_{a,j}(i) = ai + b_j$.

More specifically, $p = 6q + \zeta r$, $r = 4$, $S_K^3(p) \cong \varepsilon\mathbb{T}(p/q)$, $q = 4l + s$, $\zeta, \varepsilon \in \{1, -1\}$, $s \in \{1, 3\}$. For $\phi_{a,j}$, $\phi_{a,j}(0) = \frac{jp+q-1}{2}$, $\phi_{a,j}(\frac{p}{2}) = \frac{(1-j)p+q-1}{2}$.

Using (2) and (12), we get

$$\begin{aligned} d(L(p, 1), 0) - d(L(p, 1), \frac{p}{2}) &= \frac{p}{4} \\ d(L(p, q), \frac{q-1}{2}) - d(L(p, q), \frac{p+q-1}{2}) \\ &= \frac{p^2}{4pq} - d(L(q, \frac{1-\zeta}{2}q + \zeta r), \frac{q-1}{2}) + d(L(q, \frac{1-\zeta}{2}q + \zeta r), \frac{q+\zeta r-1}{2}) \\ &= \frac{p^2}{4pq} - \zeta \left[\frac{r^2}{4qr} - d(L(4, s), 2l + \frac{s-1}{2} \pmod{4}) + d(L(4, s), 2l + 2 + \frac{s-1}{2} \pmod{4}) \right] \\ &= \frac{3}{2} + \zeta(-1)^l \end{aligned}$$

Here we require $q - r > r$ and $\frac{q+r-1}{2} < q$, it suffices to take $p > 52 = 6 * 2r + r$.

Using Proposition 2.1, (8) and (11), we get

$$\Delta_{a,b_j}^\varepsilon(0) - \Delta_{a,b_j}^\varepsilon(\frac{p}{2}) = \frac{p}{4} \mp \varepsilon \left(\frac{3}{2} + \zeta(-1)^l - 2 \right) = 6l + \zeta \mp \varepsilon \zeta(-1)^l + \frac{3}{2}(s \mp \varepsilon) \pm 2\varepsilon$$

The parity of $\Delta_{a,b_j}^\varepsilon(0) - \Delta_{a,b_j}^\varepsilon(\frac{p}{2})$ depends only on the parity of $\frac{3}{2}(s \mp \varepsilon)$, and by (11), it should be even, so we get

$$b = \begin{cases} \frac{q-1}{2} & \text{if } s = 1, \varepsilon = 1 \quad \text{or} \quad s = 3, \varepsilon = -1 \\ \frac{p+q-1}{2} & \text{if } s = 1, \varepsilon = -1 \quad \text{or} \quad s = 3, \varepsilon = 1 \end{cases}$$

We can write $b = \frac{\bar{\theta}(\frac{s+\varepsilon}{2})p+q-1}{2}$ for $p > 52$. □

Because of Lemma 4.3, we can treat **T**-, **O**- and **I**-type manifolds uniformly. Let

$$\theta = \theta(q, \varepsilon) = \begin{cases} \bar{\theta}(q) & \text{if } r = 3, 5 \\ \bar{\theta}(\frac{s+\varepsilon}{2}) & \text{if } r = 4, q = 4l + s \end{cases},$$

then $b = \frac{\theta p+q-1}{2}$, we may denote $\phi_{a,b}$ by $\phi_{a,\theta}$.

Lemma 4.4. *Assume that $S_K^3(p) \cong \varepsilon\mathbb{T}(p/q)$. Let $m \in \{0, 1, 2, 3\}$ satisfy that*

$$0 \leq a - mq + \frac{\theta\zeta r + q - 1}{2} < q,$$

then

$$|a - \frac{mp}{6}| < \sqrt{\frac{11rp}{6}}.$$

Proof. By (7), $\Delta_{a,\theta}^\varepsilon(0) - \Delta_{a,\theta}^\varepsilon(1) = 0$ or 2. Let $h = \begin{cases} 0 & \text{if } 0 \leq \frac{\theta p + q - 1}{2} + a < p \\ 1 & \text{if } \frac{\theta p + q - 1}{2} + a \geq p \end{cases}$.

$$\begin{aligned}
 (13) \quad & \Delta_{a,\theta}^\varepsilon(0) - \Delta_{a,\theta}^\varepsilon(1) \\
 &= d(L(p, 1), 0) - \varepsilon[d(L(p, q), \frac{\theta p + q - 1}{2}) - 2\chi_{[0,q]}(\frac{\theta p + q - 1}{2})] \\
 &\quad - d(L(p, 1), 1) + \varepsilon[d(L(p, q), \frac{\theta p + q - 1}{2} + a) - 2\chi_{[0,q]}(\frac{\theta p + q - 1}{2} + a - hp)] \\
 &= 2\varepsilon\{\chi_{[0,q]}(\frac{\theta p + q - 1}{2}) - \chi_{[0,q]}(\frac{\theta p + q - 1}{2} + a - hp)\} + \frac{p^2}{4p} - \frac{(p-2)^2}{4p} \\
 &\quad - \varepsilon\{\frac{[(\theta-1)p]^2 - pq}{4pq} - d(L(q, \frac{1-\zeta}{2}q + \zeta r), \frac{\theta\zeta r + q - 1}{2}) \\
 &\quad - \frac{[2a + (\theta-1)p]^2 - pq}{4pq} + d(L(q, \frac{1-\zeta}{2}q + \zeta r), \frac{\theta\zeta r + q - 1}{2} + a - mq)\}
 \end{aligned}$$

Let $i = \frac{\theta\zeta r + q - 1}{2} \bmod r, j = \frac{\theta\zeta r + q - 1}{2} + a - mq \bmod r$. Since $0 \leq \frac{\theta\zeta r + q - 1}{2} + a - mq < q$, we use (12), the right hand side of (13) becomes

$$\begin{aligned}
 & 2\varepsilon\left(\chi_{[0,q]}(\frac{\theta p + q - 1}{2}) - \chi_{[0,q]}(\frac{\theta p + q - 1}{2} + a - hp)\right) + \frac{p-1}{p} + \varepsilon\frac{a[a + (\theta-1)p]}{pq} \\
 & + \varepsilon\zeta\left(\frac{[(\theta-1)\zeta r]^2 - qr}{4qr} - d(L(r, s), i) - \frac{[2a - 2mq + (\theta-1)\zeta r]^2 - qr}{4qr} + d(L(r, s), j)\right) \\
 & = C + \varepsilon\frac{a[a + (\theta-1)p]}{pq} - \varepsilon\zeta\frac{[a - mq + (\theta-1)\zeta r](a - mq)}{qr} \\
 & = -\frac{6\varepsilon\zeta}{pr}(a - \frac{mp}{6})^2 - \varepsilon m(1 - \theta) + \frac{\varepsilon m^2}{6} + C,
 \end{aligned}$$

where

$$C = 2\varepsilon\{\chi_{[0,q]}(\frac{\theta p + q - 1}{2}) - \chi_{[0,q]}(\frac{\theta p + q - 1}{2} + a - hp)\} + \varepsilon\zeta[d(L(r, s), j) - d(L(r, s), i)] + \frac{p-1}{p}.$$

Using (2),(3),(4), $|C| \leq \frac{6}{5} + 2 + 1 < \frac{9}{2}$.

Moreover, $|\varepsilon m(1 - \theta) + \frac{\varepsilon m^2}{6}| \leq m + \frac{m^2}{6} \leq 3 + \frac{3}{2} = \frac{9}{2}$. So we get

$$|\frac{6}{pr}(a - \frac{mp}{6})^2| < 2 + \frac{9}{2} + \frac{9}{2} = 11,$$

so our conclusion holds. \square

Lemma 4.5. Suppose $p > 767$. Let k be an integer satisfying

$$(14) \quad 0 \leq k < \frac{1}{6} \left(\frac{\sqrt{6}}{13\sqrt{11r}} \sqrt{p} - 1 \right).$$

Let

$$i_k = \frac{\theta\zeta r + q - 1}{2} + 6ka - kmp \bmod r, j_k = \frac{\theta\zeta r + q - 1}{2} + (6k+1)a - kmp - mq \bmod r.$$

Then $\Delta_{a,\theta}^\varepsilon(6k) - \Delta_{a,\theta}^\varepsilon(6k+1) = Ak + B + C_k$, where

$$\begin{aligned} A &= -\frac{72\varepsilon\zeta}{pr}\left(a - \frac{mp}{6}\right)^2 - \frac{12}{p}, \\ B &= \varepsilon\left(-\frac{6\zeta}{pr}\left(a - \frac{mp}{6}\right)^2 - m(1-\theta) + \frac{m^2}{6}\right) \\ &\quad + 2\varepsilon\{2\varepsilon\{\chi_{[0,q]}(3\theta q) - \chi_{[0,q]}((3\theta+m-6h)q)\} + \frac{p-1}{p}, \\ C_k &= \varepsilon\zeta[d(L(r,s), j_k) - d(L(r,s), i_k)]. \end{aligned}$$

and

$$h = \begin{cases} 0 & \text{if } 0 \leq 3\theta + m < 6 \\ 1 & \text{if } 3\theta + m = 6 \end{cases}.$$

Proof. Using(10), we get

(15)

$$\begin{aligned} &\Delta_{a,\theta}^\varepsilon(6k) - \Delta_{a,\theta}^\varepsilon(6k+1) \\ &= d(L(p,1), 6k) - d(L(p,1), 6k+1) - \varepsilon[d(L(p,q), \frac{\theta p + q - 1}{2} + 6ka - kmp) \\ &\quad - 2\chi_{[0,q]}(\frac{\theta p + q - 1}{2} + 6ka - kmp)] + \varepsilon[d(L(p,q), \frac{\theta p + q - 1}{2} + (6k+1)a - kmp) \\ &\quad - 2\chi_{[0,q]}(\frac{\theta p + q - 1}{2} + (6k+1)a - (km+h)p)] \\ &= 2\varepsilon\{\chi_{[0,q]}(\frac{\theta p + q - 1}{2} + 6ka - kmp) - \chi_{[0,q]}(\frac{\theta p + q - 1}{2} + (6k+1)a - (km+h)p)\} \\ &\quad + \frac{(p-12k)^2}{4p} - \frac{[p-2(6k+1)]^2}{4p} - \varepsilon\left\{\frac{[12ka - (2km+1-\theta)p]^2}{4pq} \right. \\ &\quad \left. - \frac{[(12k+2)a - (2km+1-\theta)p]^2}{4pq} - d(L(q, \frac{1-\zeta}{2}q + \zeta r), \frac{\theta\zeta r + q - 1}{2} + 6ka - kmp) \right. \\ &\quad \left. + d(L(q, \frac{1-\zeta}{2}q + \zeta r), \frac{\theta\zeta r + q - 1}{2} + (6k+1)a - kmp - mq)\right\} \end{aligned}$$

We require

$$\begin{aligned} 0 &\leq \frac{\theta\zeta r + q - 1}{2} + 6ka - kmp < q, \\ 0 &\leq \frac{\theta\zeta r + q - 1}{2} + (6k+1)a - kmp - mq < q. \end{aligned}$$

It suffices that

$$k < \frac{1}{6} \left(\frac{q-9}{2} \sqrt{\frac{6}{11rp}} - 1 \right).$$

This implies

$$(16) \quad 3\theta q \leq \frac{\theta p + q - 1}{2} + 6ka - kmp < (3\theta + 1)q,$$

$$(17) \quad (3\theta + m)q \leq \frac{\theta p + q - 1}{2} + (6k+1)a - kmp < (3\theta + m + 1)q.$$

When $m = 3$, $\theta = 1$, (17) becomes

$$6q \leq \frac{\theta p + q - 1}{2} + (6k + 1)a - kmp < 7q.$$

Here we require

$$p \leq \frac{\theta p + q - 1}{2} + (6k + 1)a - kmp < p + q.$$

We know $a < \frac{p}{2}$, so

$$\frac{\theta p + q - 1}{2} + (6k + 1)a - kmp < \frac{\theta p + q - 1}{2} + \frac{p}{2} < p + q.$$

Moreover, we know when $m = 3$, by Lemma 4.4, $a > \frac{p}{2} - \sqrt{\frac{11rp}{6}}$. If

$$k \leq \frac{1}{6} \left(\frac{q-1}{2} \sqrt{\frac{6}{11rp}} - 1 \right),$$

then

$$\frac{\theta p + q - 1}{2} + (6k + 1)a - kmp \geq p.$$

When $p > 767$,

$$\frac{1}{6} \left(\frac{p}{13} \sqrt{\frac{6}{11rp}} - 1 \right) < \frac{1}{6} \left(\frac{q-9}{2} \sqrt{\frac{6}{11rp}} - 1 \right).$$

Using (12), (16) and (17), the right hand side of (15) becomes

$$\begin{aligned} & 2\varepsilon\{\chi_{[0,q]}(3\theta q) - \chi_{[0,q]}((3\theta + m - 6h)q)\} + \frac{p - (12k + 1)}{p} \\ & + \varepsilon \frac{a((12k + 1)a - 2kmp + (\theta - 1)p)}{pq} \\ & + \varepsilon \zeta \left\{ \frac{[12ka - 2kmp + (\theta - 1)\zeta r]^2 - qr}{4qr} - d(L(r, s), i_k) \right. \\ & \left. - \frac{[2(6k + 1)a - 2kmp - 2mq + (\theta - 1)\zeta r]^2 - qr}{4qr} + d(L(r, s), j_k) \right\}. \end{aligned}$$

This simplifies to be

$$\begin{aligned} & 2\varepsilon\{\chi_{[0,q]}(3\theta q) - \chi_{[0,q]}((3\theta + m - 6h)q)\} + \varepsilon \zeta [d(L(r, s), j_k) - d(L(r, s), i_k)] + \frac{p - (12k + 1)}{p} \\ & + \varepsilon \frac{a((12k + 1)a - 2kmp + (\theta - 1)p)}{pq} - \varepsilon \zeta \frac{((12k + 1)a - 2kmp - mq + (\theta - 1)\zeta r)(a - mq)}{qr} \\ & = - \frac{6(12k + 1)\varepsilon \zeta}{pr} \left(a - \frac{mp}{6} \right)^2 - \varepsilon m(1 - \theta) + \frac{\varepsilon m^2}{6} + 2\varepsilon\{\chi_{[0,q]}(3\theta q) - \chi_{[0,q]}((3\theta + m - 6h)q)\} \\ & + \varepsilon \zeta [d(L(r, s), j_k) - d(L(r, s), i_k)] + \frac{p - (12k + 1)}{p} \\ & = Ak + B + C_k. \end{aligned}$$

□

Proof of Proposition 4.1. If $S_K^3(p) \cong \varepsilon \mathbb{T}(p/q)$, then (11) holds, so

$$(18) \quad \Delta_{a,\theta}^\varepsilon(6k) - \Delta_{a,\theta}^\varepsilon(6k + 1) = 0 \text{ or } 2$$

for all k satisfying (14). If $p > 310r(36r + 1)^2$, then

$$6 \cdot 6r + 1 < \frac{\sqrt{6}}{13\sqrt{11}r} \sqrt{p}$$

hence $k = 6r$ satisfies (14).

Let A, B, C_k be as in Lemma 4.4. If $A \neq 0$, then $Ak + B + C$ is equal to 0 or 2 for at most two values of k for any given C . Given $p, q, a, \varepsilon, \zeta$, as k varies, C_k can take at most $3r$ values. It follows that $Ak + B + C_k$ cannot be 0 or 2 for $k = 0, 1, \dots, 6r$. As a consequence, if $p > 310r(36r + 1)^2$, then (18) does not hold.

The only case we need to consider is that $A = 0$. In this case $\varepsilon\zeta = -1$.

$$A = \frac{12}{p} \left(\frac{6}{r} \left(a - \frac{mp}{6} \right)^2 - 1 \right) = 0.$$

We get $|a - \frac{mp}{6}| = \sqrt{\frac{r}{6}}$, which is an irrational number. This contradicts that a is an integer and $\frac{mp}{6}$ is a rational number. \square

Since we get an upper bound for p , an easy computer search will yield all possible p/q 's. They are $1/1, 2/1, 3/1, 7/2, 9/1, 9/2, 10/1, 10/1, 11/1, 13/3, 13/3, 14/3, 17/2, 17/2, 19/4, 21/4, 22/3, 23/3, 27/4, 27/5, 29/4, 29/4, 37/7, 37/7, 38/7, 43/8, 46/7, 47/7, 49/9, 50/9, 51/8, 58/9, 59/9, 62/11, 69/11, 70/11, 81/13, 81/14, 83/13, 86/15, 91/16, 93/16, 94/15, 99/16, 99/17, 101/16, 106/17, 106/17, 110/19, 110/19, 113/18, 113/18, 119/19, 131/21, 133/23, 137/22, 137/22, 143/23, 146/25, 154/25, 157/27, 157/27, 163/28, 211/36, 221/36$. Here if p_i/q_i appears twice, this means they correspond to candidate knots with different Heegaard Floer Homologies.

5. BERGE KNOTS WITH **T**-, **O**- AND **I**-TYPE SURGERIES

Proposition 5.1. *Only 11 hyperbolic berge knots have **T**-, **O**- and **I**-type surgeries. More precisely, let $K(p, q; \lambda)$ be the berge knot corresponding to homology class λ in $L(p, q)$. They are $K(18, 5; 5)$, $K(39, 16; 16)$, $K(45, 19; 8)$, $K(46, 19; 11)$, $K(68, 19; 5)$, $K(71, 27; 11)$, $K(82, 23; 5)$, $K(93, 26; 5)$, $K(107, 30; 5)$, $K(118, 33; 5)$, $K(132, 37; 5)$.*

Proof. Berge knots have lens space surgeries, we can compute their Heegaard Floer Homology using Proposition 2.1 and compare them with the list from finite surgeries, we get 11 candidates. We draw the link diagram in SnapPy and compute the fundamental group of the Dehn filling with the finite surgery coefficient. If what we get is a $(2, 3, n)$ -type group, then we have verified that the knot has indeed a required finite surgery. Below is a table of the candidates.

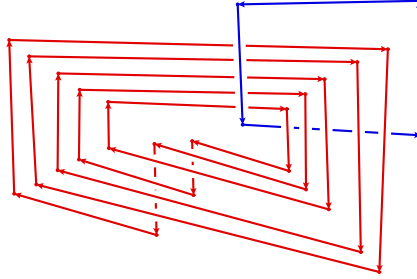
Here we use a point of view of dual berge knots in the corresponding lens space, as they have the same knot complement as berge knots in S^3 , the only thing is to figure out the corresponding coefficients. The computation is as follows.

We would like to draw a link L with two components as above, and denote the link complement by M . When we perform on the trivial component (1-component) p/q -surgery, we get $L(p, q)$. The other component then becomes the dual berge knot in $L(p, q)$. We would like to choose the orientations of two components consistently, one choice is shown above, and the other choice is to reverse the orientations of both two components. Denote the longitude and meridian of 0-component by λ and μ , and those of 1-component by l and m . In homology, we have $w\mu = l$ and $\lambda = wm$. After performing p/q -surgery on 1-component, we have $pm + ql = 0$ in homology and we denote the longitude and meridian of resultant 0-component K' by λ' and μ' . We

TABLE 1. Canditaes of Berge knots with **T**-,**O**- and **I**-type surgeries

p	q	λ	finite surgery coefficient p'
18	5	5	17
39	16	16	38
45	19	8	46
46	17	11	47
68	19	5	69
71	21	11	70
82	23	5	81
93	25	5	94
107	25	5	106
118	25	5	119
132	25	5	131

0 1

FIGURE 1. Link for $K(107, 30; 5)$

have $qw^2\mu + p\lambda = qwl + pwm = 0$, this means that $\lambda' = (qw^2, p)$. By performing p -surgery on K' we get $L(p, q)$ (this corresponds to performing ∞ -surgery on 0-component of L), and by performing ∞ -surgery on K' we get S^3 , and we hope to get a spherical space form by performing p' -surgery on K' . We have $\lambda' = qw^2\mu + p\lambda = (qw^2, p)$, and $p\mu' + \lambda' = (\pm 1, 0)$. Note that $L(p, q_1) \cong L(p, q_2)$ iff $q_1q_2 \equiv \pm 1 \pmod p$, so there is an indeterminacy in q . Take $K(107, 25; 5)$ for example. We have $\lambda' = (25 \cdot 5^2, 107)$, and $107\mu' + \lambda' = (\pm 1, 0)$. There is no integer solutions. Instead we take $q = 30$, as $\text{mod}(30 \cdot 25, 107) = 1$, denote the knot by $K(107, 30; 5)$ from now on. We thus have $\lambda' = (30 \cdot 5^2, 107)$, $\mu' = (-7, -1)$, and $107\mu' + \lambda' = (1, 0)$. $M(1/0, 107/30)$ is $L(107, 30)$, and $M(7/1, 107/30)$ is S^3 . And $M(8/1, 107/30)$ has fundamental group of the form $\langle a, b | abab^{-2}, ab(b^3a^2)^{77}b^4ab((a^{-2}b^{-3})^3a^{-1})^{21}a^{-1}b \rangle$. The first relator is $(ab)^2 = b^3$, which is in the center. Mod it out, the other relator becomes $a^4 = 1$. By upper central series theory, this group is finite and is of type $(2, 3, 4)$, which means this dehn filling is indeed the required finite surgery. \square

TABLE 2. Berge knots with **T**-, **O**- and **I**-type surgeries

p	q	λ	finite surgery coefficient p'	$Z(\pi_1(S_K^3(p')))$	$\pi_1(S_K^3(p'))/Z(\pi_1(S_K^3(p')))$
18	5	5	17	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = b^3 = a^5 = 1 \rangle$
39	16	16	38	$\langle a^4 \rangle$	$\langle a, b b^2 = (a^{-1}b)^3 = a^4 = 1 \rangle$
45	19	8	46	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = a^3 = b^4 = 1 \rangle$
46	17	11	47	$\langle b^3 \rangle$	$\langle (b^2a^{-2})^2 = b^3 = a^5 = 1 \rangle$
68	19	5	69	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = b^3 = a^3 = 1 \rangle$
71	27	11	70	$\langle a^3 \rangle$	$\langle a, b b^2 = a^3 = (ba^2)^4 = 1 \rangle$
82	23	5	81	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = b^3 = a^3 = 1 \rangle$
93	26	5	94	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = b^3 = a^4 = 1 \rangle$
107	30	5	106	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = b^3 = a^4 = 1 \rangle$
118	33	5	119	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = b^3 = a^5 = 1 \rangle$
132	37	5	131	$\langle (ab)^2 \rangle$	$\langle a, b (ab)^2 = b^3 = a^5 = 1 \rangle$

6. SUMMARY

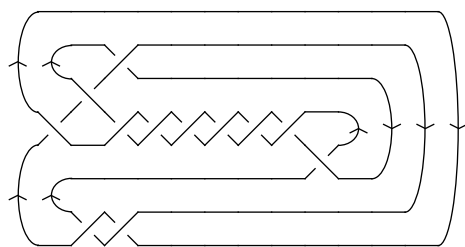
I summarize all results below with a table of torus knots and satellite knots and a table of hyperbolic knots. Let $T(p, q)$ be the (p, q) torus knot, and let $[p_1, q_1; p_2, q_2]$ denote the (p_1, q_1) cable of $T(p_2, q_2)$. I have also drawn all hyperbolic knots below using Mathematica package KnotTheory`.

TABLE 3. torus knots and satellite knots with **T**-, **O**- and **I**-type surgeries

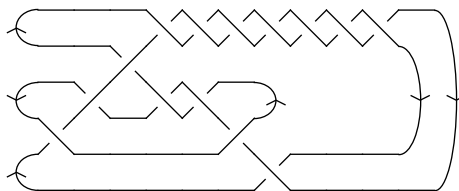
p	knot	p	knot	p	knot	p	knot	p	knot
1	$T(3, 2)$	2	$T(3, 2)$	3	$T(3, 2)$	7	$T(5, 2)$	9	$T(3, 2)$
10	$T(3, 2)$	10	$T(4, 3)$	11	$T(3, 2)$	13	$T(5, 3)$	13	$T(5, 2)$
14	$T(4, 3)$	17	$T(5, 3)$	19	$[9, 2; 3, 2]$	21	$[11, 2; 3, 2]$	27	$[13, 2; 3, 2]$
29	$[15, 2; 3, 2]$	37	$[19, 2; 5, 2]$	43	$[21, 2; 5, 2]$	49	$[16, 3; 3, 2]$	50	$[17, 3; 3, 2]$
59	$[20, 3; 3, 2]$	91	$[23, 4; 3, 2]$	93	$[23, 4; 3, 2]$	99	$[25, 4; 3, 2]$	101	$[25, 4; 3, 2]$
106	$[35, 3; 4, 3]$	110	$[37, 3; 4, 3]$	133	$[44, 3; 5, 3]$	137	$[46, 3; 5, 3]$	146	$[29, 5; 3, 2]$
154	$[31, 5; 3, 2]$	157	$[39, 4; 5, 2]$	163	$[41, 4; 5, 2]$	211	$[35, 6; 3, 2]$	221	$[37, 6; 3, 2]$

TABLE 4. hyperbolic knots with **T**-, **O**- and **I**-type surgeries

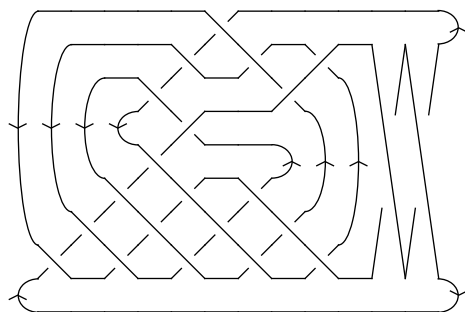
p	knot
17	Preztel knot $P(-2, 3, 7)$
22	Preztel knot $P(-2, 3, 9)$
23	Preztel knot $P(-2, 3, 9)$
29	mirror image of $K(1, 1, 0)$ from section 4 of [8]
37	$K(11, 3, 2, 1, 1)$ from [6]
38	Berge knot $K(39, 16; 16)$
46	Berge knot $K(45, 19; 8)$
47	K_3^* from [4]
51	K_2^\sharp from [4]
58	mirror image of the P/SF_d KIST IV knot with $(n, p, \epsilon, J_1, J_2) = (-2, 1, 1, -4, 1)$ from [2]
62	P/SF_d KIST III knot with $(h, k, h', k', J) = (-5, -3, -2, -1, 1)$ from [2]
69	mirror image of $K(2, 3, 5, 1, -3)$ from [12]
70	Berge knot $K(71, 27; 11)$
81	$K(2, 3, 5, 1, 3)$ from [12]
83	P/SF_d KIST V knot with $(n, p, \epsilon, J_1, J_2) = (1, -2, -1, 2, 2)$ from [2]
86	mirror image of $K(3, 4, 7, 1, -2)$ from [12]
94	mirror image of $K(2, 3, 5, 1, -4)$ from [12]
106	$K(2, 3, 5, 1, 4)$ from [12]
110	$K(3, 4, 7, 1, 2)$ from [12]
113	mirror image of $K(3, 5, 8, 1, -2)$ from [12]
113	mirror image of the P/SF_d KIST V knot with $(n, p, \epsilon, J_1, J_2) = (-3, -2, -1, 2, 2)$ from [2]
119	mirror image of $K(2, 3, 5, 1, -5)$ from [12]
131	$K(2, 3, 5, 1, 5)$ from [12]
137	mirror image of $K(2, 5, 7, 1, -3)$ from [12]
143	$K(3, 5, 8, 1, 2)$ from [12]
157	$K(2, 5, 7, 1, 3)$ from [12]



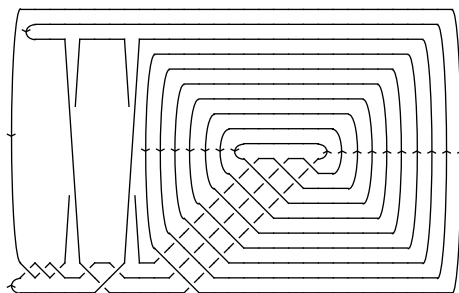
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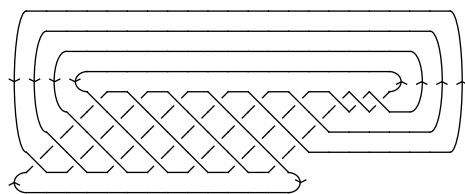
22,23



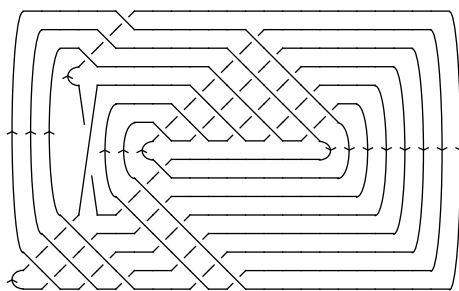
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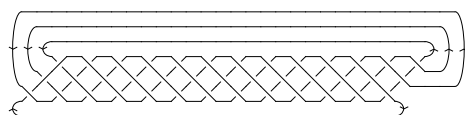
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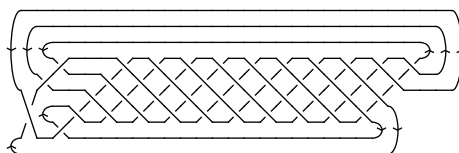
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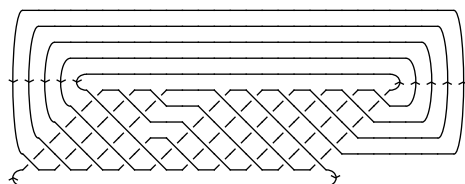
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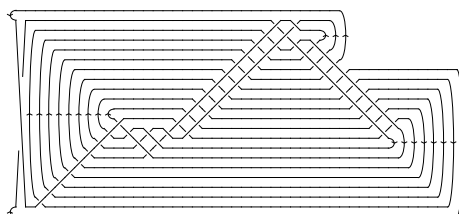
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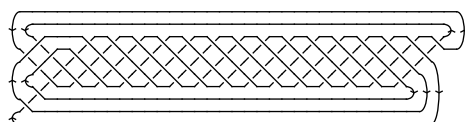
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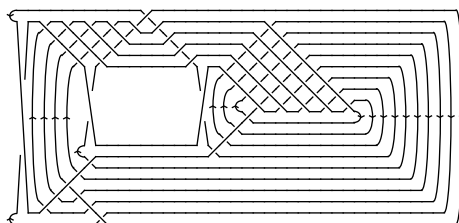
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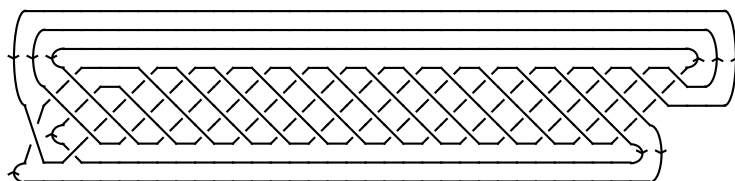
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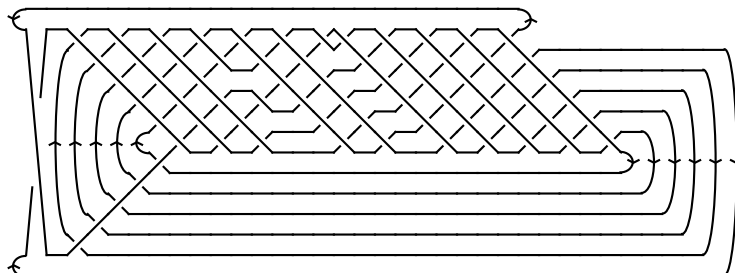
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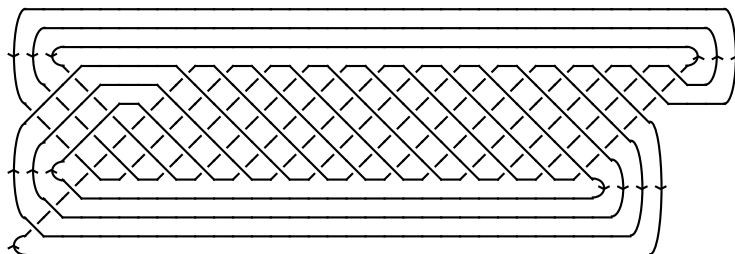
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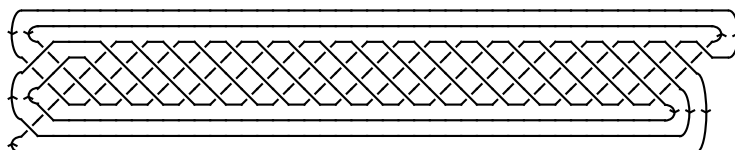
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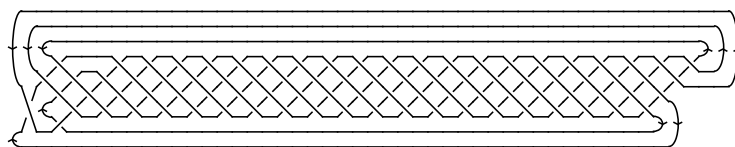
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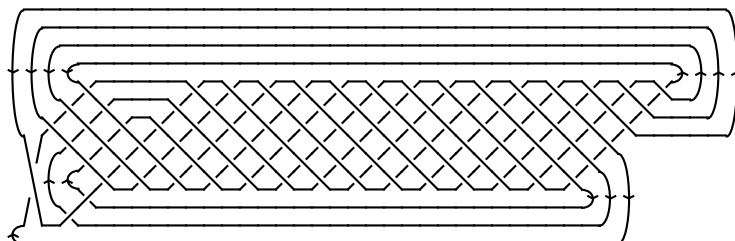
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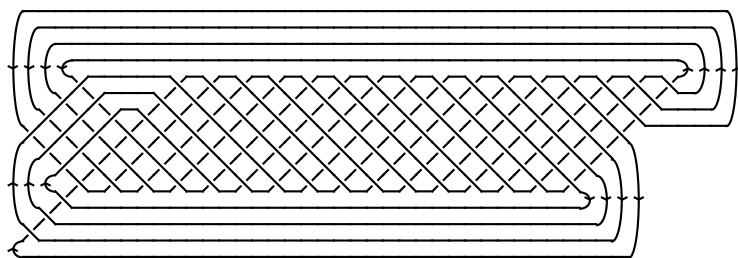
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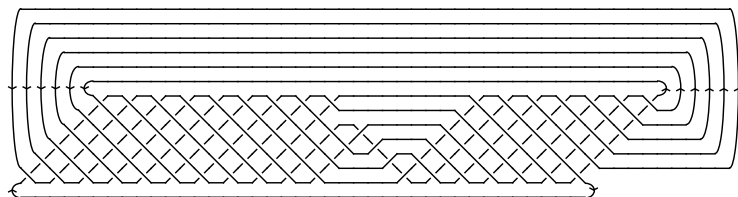
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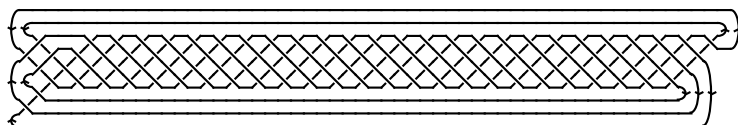
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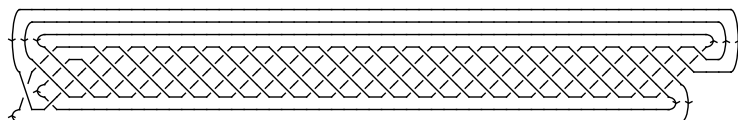
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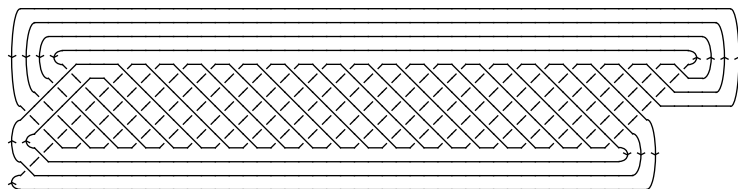
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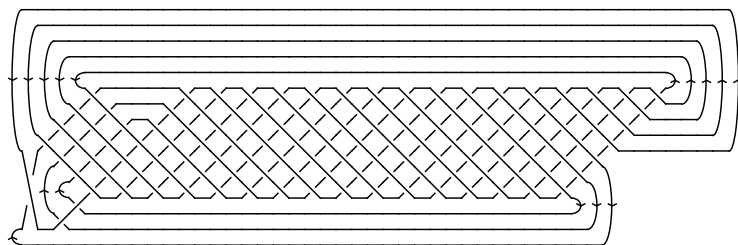
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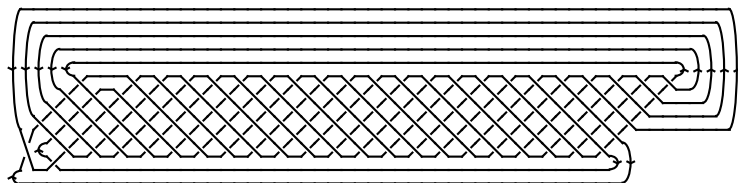
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143



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DEPARTMENT OF MATHEMATICS, CALTECH, 1200 E CALIFORNIA BLVD, PASADENA, CA 91125
E-mail address: g11007@caltech.edu